

The role of quantum correlations in Cop and Robber game

Adam Glos and Jarosław Adam Miszcza

Institute of Theoretical and Applied Informatics, Polish Academy of Sciences,
Bałtycka 5, 44-100 Gliwice, Poland

25/02/2017 (v. 1.22)

Abstract

We introduce and study quantized versions of Cop and Robber game. We achieve this by using graph-preserving unitary operations, which are the quantum analogue of stochastic operations preserving the graph. We provide the tight bound for the number of operations required to reach the given state. By extending them to controlled operations, we define a quantum controlled Cop and Robber game, which expands the classical Cop and Robber game, as well as classically controlled quantum Cop and Robber game. In contrast to the typical scheme for introducing quantum games, we assume that both parties can utilise full information about the opponent's strategy. We show that the utilisation of the full knowledge about the opponent's state does not provide the advantage. Moreover, the chances of catching the Robber decreases for classically cop-win graphs. The result does not depend on the chosen model of evolution. On the other hand, the possibility to execute controlled quantum operations allows catching the Robber on almost all classically cop-win graphs. To provide interesting, non-trivial quantized Cop and Robber game, we need to enrich the structure of correlations between the players' systems. This result demonstrates that the ability to utilise quantum controlled operations is significantly stronger than the control restricted operating on classical selecting quantum operations only.

Keywords: combinatorial games, quantum walks, quantum correlations.

MSC classes: 05C57 (Primary), 91A46, 81P40 (Secondary).

1 Introduction

Quantum information processing in complex networks is based on the assumption that parties (or agents) acting in the network can utilize quantum carriers of information for the purpose of controlling the execution of protocols or algorithms. In this scenario, it is reasonable to assume that the integrity of protocol execution should be secure even against the attacker possessing the ability to operate on quantum data. In other words, one has to revise the results concerning the security of classical distributed protocols taking into account the quantum model of computation [1].

Suppose we have a fixed reflexive graph G and two players. We define a game as follows: the first player (usually called *Cop* or *pursuer*) chooses a vertex on which he starts and then the second player called *Robber* or *Evader*, chooses his vertex. After that, each player changes their position to neighbouring vertex sequentially. If, after a finite number of rounds, the Cop and Robber ended in the same vertex, we say that the Cop wins the game. Otherwise, we say, that Robber is the winner.

The game called *Cop and Robber game* was firstly analysed by Quilliot [2] and Nowakowski and Winkler [3]. The first author provided a beautiful characterization of finite copwin graphs, *i.e.* graphs on which the Cop has a winning strategy, and the other authors extended it to the case of infinite graphs. Since then, many of variants of the game have been proposed, including the game where players do not see each other [4], the players move simultaneously [5], the Cop plays heuristically [6, 7], the play is continuous in time [8, 9]. The game provides many hard mathematical problems, where the best-known are the characterization of multi-copwin graph [10] and Meyniel conjecture [11]. Furthermore, the game provides an interesting algorithmic problem: demonstrating that a graph needs at most k Cops for winning the game is NP-hard. The game has also found its applications in robotics [12], modelling graph searching [13], and analysing the security in complex networks [14]. For a survey of the game and its properties see [15].

In this work, we introduce the quantum games based on the assumption that both parties possess full information about the strategy of the second party. In quantum game theory, this assumption is rare as the main motivation to study quantum games is to extend the space of possible moves [16], utilise entanglement for the synchronization of the moves [17], or to share a quantum state for providing means for cooperation [18]. The first example of a quantum game was provided in the form of simple penny flip game [19]. In this scenario, a player with the ability to use quantum strategies (moves) can always win with a player using only classical moves. The most important result in this area is based on the possibility to use shared entangled state for cooperating in the Prisoner's Dilemma. Such an approach was used in [20] to analyse quantum Prisoner's Dilemma as a pursuit game on a graph, where the Cops utilise a shared entangled state.

In contrast, this paper provides the contribution in the field of combinatorial games, where all players (or parties) possess full knowledge of the state of the system. In the case of quantum combinatorial games, this is to say that both players possess the full knowledge about the previous moves of the opponents. In other words, we are interested in the existence of strategies, which are optimal in some sense. The properties of such games defined in quantum information systems were previously studied in the context of network exploration problems [21] or modelling trapping in quantum networks [22]. Here we consider the possibility to define a quantum extension of the Cop and Robber game, which describes the protocol for confining the possibility of moves by one of the parties. The main question arising in this context is the possibility of constructing a strategy for the Cop player to catch the Robber, taking into account the probabilistic nature of quantum measurement. We analyse this problem by introducing suitable winning conditions.

The main contribution of the presenter paper is obtaining a nontrivial quantum version of the game. We propose and investigate graph-preserving unitary evolution, which can be used in extending other graph-based games *e.g.* simple stochastic games [23]. We prove the tight bound for the number of required operations. Moreover, we show that quantum controlled gates, and hence quantum entanglement, provide new strategies which diametrically change the course of the game. Hence, we provide an argument confirming the crucial role of entanglement in quantum information processing.

The paper is organised as follows. In Section 2, we introduce mathematical apparatus, including concepts necessary to deal with quantum measurement. We achieve this by introducing classical game called an open probabilistic Cop and Robber game. We show that the game trivialises in the sense of the winning strategy. In Section 3 we define graph-preserving unitary evolution and by applying it we introduce classical controlled quantum Cop and Robber game. We show that such a game cannot be used to obtain any advantage in the sense of the wider class of winning strategies. We also provide a tight bound for state obtainability. In Section 4, we extend the graph-preserving unitary evolution to describe the non-trivial extension of the game, namely quantum controlled Cop and Robber game. We demonstrate that such version provides an advantage over the classically controlled quantum scheme. Finally, in Section 5, we provide a summary of the presented results.

2 Preliminaries

We start by introducing mathematical apparatus required for the analysis of the quantum Cop and Robber game. In particular, we provide some facts related to the probabilistic version of the game, which give some insight into the behaviour of the players in the quantum realm.

2.1 Terminology

Suppose we have a digraph G with nodes set V and arcs $E \subset V \times V$. We say that the vertex $v \in V$ is a neighbour of $u \in V$ iff $(u, v) \in E$. The set of all neighbours of v is denoted by $N(v)$. A graph is called reflexive, iff for each $v \in V$ we have $(v, v) \in E$. The reflexive graphs are the only one considered here. We say that the vertex v is a corner if there exists the vertex u such that $N(v) \subset N(u)$. A spanning tree is a sub-graph that is a tree which includes all of the vertices of G . We call $A \subset V$ a dominating set iff for an arbitrary vertex $v \in V$ we have $v \in A$ or $N(v) \cap A \neq \emptyset$.

A graph homomorphism from a graph G to G' is an edge-preserving mapping from V to V' . We call a graph homomorphism f a retraction if for each $v' \in V'$ we have $f(v') = v'$. Then we call G' a retract of G and denote it by $G \upharpoonright V'$.

2.2 Open probabilistic Cop and Robber game

As it has already been mentioned, many variations of the game can be proposed. One of the most popular versions is the ‘Hunter and rabbit’ game [4], where players do not

see each other until they are at the same vertex. In that case, the strategy needs to be described with the stochastic operations and the players' positions are described with probability vector. Note that the probabilistic version of the Cop and Robber game is no longer open since both the position of the player, even in the sense of probability vector, and the performed stochastic operation are unknown to the opponent.

In order to introduce open probabilistic version of the Cop and Robber game we use the following scheme. Suppose that the graph is known to both players. The Cop and the Robber, in that order, choose their initial position randomly, *i.e.* they select the probabilistic vector of a position. Next, they sequentially perform stochastic operations preserving the graph structure on their state. In each step, both the Cop and the Robber do not know their position and their opponent's position but they know the current probability of a player being in all vertices. The Cop chooses when he decides to perform a measurement, that is when he uncovers the board.

In order to consider probabilistic versions of the Cop and Robber, including the variant with quantum strategies, we need to take into account the probabilistic nature of quantum measurement. To this end we define the following classes of winning conditions.

Definition 1. *We say that the graph is p -copwin, if there is such strategy for the Cop that after finite number of steps the probability of measuring the players in the same vertex is greater than p .*

Definition 2. *We say that the graph is nearly p -copwin, if for arbitrary $\varepsilon > 0$ there is a strategy for the Cop such that he can win with probability at least $p - \varepsilon$.*

It is worth noting that in the sense of the strategy set, the probabilistic version expands the original, deterministic Cop and Robber game. Every move in the original one can be described as a stochastic operation. When we restrict players to deterministic probability vectors, each course of the original game can be described with open probabilistic Cop and Robber formalism.

However, it is easy to see that this model narrows the set of feasible strategies. To demonstrate this let us suppose that graph $G = (V, E)$ is given. The Cop chooses as initial state vector $p_C(v) = \frac{1}{|V|}$. For arbitrary Robber's position p_R the probability of measuring players in the same vertex equals

$$p_{\text{copwin}} = \sum_{v \in V} p_C(v) p_R(v) = \frac{1}{|V|} \sum_{v \in V} p_R(v) = \frac{1}{|V|}. \quad (1)$$

Note that the probability of winning with this strategy does not depend on the Robber's position. Hence, an arbitrary graph is at least $\frac{1}{|V|}$ -copwin. If the Robber chooses the same initial state we can show that the arbitrary graph is exactly $\frac{1}{|V|}$ -copwin. Note that the strategy does not depend on the edge set, and henceforth the graph does not need to be undirected or even connected. The probability depends on the size of V only.

2.3 Probabilistic versus deterministic player

Let us now analyze how the introduced scheme influences the class of cop-win graphs. To achieve this let us analyse an unfair game on cop-win graphs. It is easy to see that if the Robber is limited to deterministic strategies while the Cop is allowed to use randomised strategies, the latter can win with probability arbitrary close to one. The following theorem gives the full characterization of this situation.

Theorem 1. *In open probabilistic Cop and Robber game with the Robber using deterministic moves, the Cop can win with probability one if and only if the graph is cop-win. Otherwise, the Cop can win with the probability arbitrarily close to one.*

Proof. The sufficiency of graph being copwin comes from the fact that the Cop can use the deterministic strategy to win. In the not cop-win graph case, the Robber can always evade part of the probability of the Cop localised in some vertex, hence we have necessity.

Now we show that an arbitrary graph is nearly 1-copwin. The Cop spreads uniquely on the arbitrary dominating set of the graph. Next he can ‘catch’ the evader with probability one over the cardinality of the dominating set. The part of probability which is on the Robber’s position will now follow him, while the rest of the probability repeat the strategy. Each round needs the linear time of single steps on a number of vertices, and one can show that the probability of winning for the Robber decreases geometrically with the number of rounds. \square

If we swap the limitations, the Cop can win with probability at most $\frac{1}{|V|}$, since the Robber can choose the uniform distribution as the initial state. However, if a graph is not copwin, the Robber can use deterministic strategies to fully avoid the pursuer. We have not found the full characterization of this situation for copwin graphs

3 Graph-preserving discrete evolution

The first quantum version of the Cop and Robber game considered in this paper is defined by using quantum unitary operation, preserving the structure of a graph. In this section, we use such operations to introduce a direct quantum version of the open probabilistic Cop and Robber game. We call this game the classically controlled quantum Cop and Robber game to emphasise the fact that the introduced scheme is based on the full knowledge of the quantum state of the opponent. We show that in the context of winning strategy the game does not change comparing to the probabilistic version. We also show that for arbitrary connected, reflexive and undirected graph every quantum state is obtainable in $O(|V|)$ steps, which is tight bound. We also provide examples that finding a graph preserving discrete evolution is not always trivial and can lead to some unintuitive results.

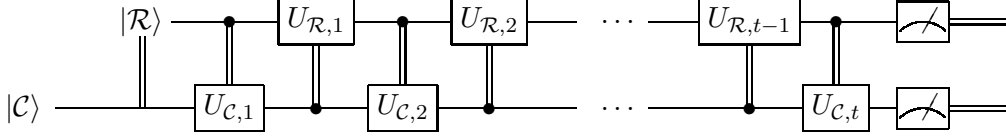


Figure 1: Classically controlled quantum Cop and Robber game. The Cop chooses his initial state first, then the Robber choose his state. Double line denotes classical control, since both parties possess the full knowledge of their and their opponent's current state. The measurement is performed in the basis $\{|v\rangle : v \in V\}$.

3.1 Direct approach

In the probabilistic version of the Cop and Robber game, the players can choose an arbitrary stochastic operations preserving the graph structure. Such operation is defined as follows.

Definition 3. We say that the stochastic operation M preserves the graph, if for arbitrary disconnected vertices v and w we have $M_{wv} = 0$.

In a similar manner, one can define quantum operation preserving graph structure.

Definition 4. We say that the unitary operation U preserves the graph, if for arbitrary disconnected vertices v and w , represented by state $|v\rangle$ and $|w\rangle$ respectively, we have $\langle w|U|v\rangle = 0$.

Using the above definition one can introduce the quantized version of Cops and Robbers game, called *classically controlled quantum Cop and Robber game*. At the beginning of this game, both players choose arbitrary pure quantum state. The evolution of pure states is described with unitary operation U such that for arbitrary disconnected vertices v and w we have $\langle w|U|v\rangle = 0$. The \mathbf{U}_G denotes the set of all unitary operations preserving the graph G . The \mathbf{U}_G contains the identity (in the case of reflexive graphs, which are the only ones considered here) and is closed under Hermitian transposition. However, it does not form a group because it is not closed under multiplication.

More precisely, we can define the game as follows. Each player has their own system \mathcal{H}_R and \mathcal{H}_C spanned by orthogonal set $\{|v\rangle : v \in V\}$ and chooses the initial quantum state $|\mathcal{R}_0\rangle$ and $|\mathcal{C}_0\rangle$. The combined system is of the form $\mathcal{H}_R \otimes \mathcal{H}_C$ and the initial state is of the form $|\mathcal{R}_0\rangle \otimes |\mathcal{C}_0\rangle$. Then, in each iteration the Cop and the Robber perform operations $I_R \otimes U_C$ and $U_R \otimes I_C$, respectively, where $U_C, U_R \in \mathbf{U}_G$. In the end, they perform the measurement in basis $\{|v\rangle : v \in V\}$. The Cop wins if, after t steps and his move, both players are measured in the same vertex. In this situation, if $|S_t\rangle$ denotes game state after t steps, then the probability of the Cop being a winner reads

$$p_{\text{copwin}} = \sum_{v \in V} |(\langle v| \otimes \langle v|) |S_t\rangle|^2. \quad (2)$$

One should note that by local operation one cannot entangle the registers. For this reason the state is of the form $|S_t\rangle = |\mathcal{R}_t\rangle \otimes |\mathcal{C}_t\rangle$ and the formula above simplifies to

$$p_{\text{copwin}} = \sum_{v \in V} |\langle v | \mathcal{R}_t \rangle \langle v | \mathcal{C}_t \rangle|^2. \quad (3)$$

Quantum circuit for the above game is presented in the Fig. 1.

The quantized version of the Cop and Robber game introduced above can be understood as a reformulation of the open probabilistic Cop and Robber game in the language of quantum computing. It can be seen that the ability to utilise local quantum moves does not provide any party with any advantage. To demonstrate this one can observe that the Cop and the Robber can choose the flat superposition of the base states, $\frac{1}{\sqrt{|V|}} \sum_{v \in V} |v\rangle$, as the initial state. It is easy to observe that every graph is $\frac{1}{\sqrt{|V|}}$ -copwin. One should note that this example suggests that the triviality do not depend even on the evolution form – similar results will be obtained if we choose as an evolution discrete quantum walk model or other models.

The unfair game yields the same results as an open probabilistic game. We can conclude the results as follows: *in order to provide interesting, non-trivial quantized Cop and Robber game, we need to enrich the structure of correlations between the players' systems.* This motivates the quantum model based on the ability of executing non-local quantum gates, developed in Section 4.

3.2 Tight bound for state obtainability

It was shown [24] that in order to evolve using operations from \mathbf{U}_G the graph may be directed but it must be reversible, *i.e.* if there is a path from v to w , then there is a path from w to v . However, it is not enough for obtaining arbitrary result state. Suppose $G = (\{0, \dots, n-1\}, E)$ is a directed, clockwise cycle. Then, the \mathbf{U}_G consist of identity operations and clockwise permutations changing at most the amplitude phase, *i.e.*

$$U = \sum_{i=0}^{n-1} e^{i\alpha_i} |i \oplus 1\rangle \langle i|, \quad (4)$$

where \oplus denotes addition modulo n , and $\alpha_i \in \mathbb{R}$. As an example, if a state $|0\rangle$ is given, it is impossible to obtain superposition $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Below we present a sufficient condition for arbitrary state obtainability.

Theorem 2. *Let $G = (V, E)$ be such a reflexive, reversible digraph that contains an undirected spanning tree. Then, for arbitrary states $|\varphi\rangle$ and $|\psi\rangle$ there exists a sequence $U_1, \dots, U_{2|V|-2} \in \mathbf{U}_G$ such that*

$$|\psi\rangle = U_{2|V|-2} \dots U_1 |\varphi\rangle. \quad (5)$$

Proof. Suppose v and w are connected vertices, and $|\varphi\rangle = \alpha_v |v\rangle + \alpha_w |w\rangle$. It is simple to show that for arbitrary $|\psi\rangle = \beta_v |v\rangle + \beta_w |w\rangle$ such that $\langle \psi | \psi \rangle = \langle \varphi | \varphi \rangle$, there exists

a unitary operation U preserving the graph such that $|\psi\rangle = U|\varphi\rangle$. This is equivalent to the fact that, if v and w are connected, then their arbitrary superposition can be changed to an arbitrary superposition with the same norm.

Let $n = |V|$. The proof goes as follows. First we show constructively that for an arbitrary state $|\varphi\rangle$ and vertex $v \in V$ we can find a sequence of unitary operations preserving the graph structure, changing $|\varphi\rangle$ to $|v\rangle$. Similarly, one can reverse the method to obtain $|\psi\rangle$ from $|v\rangle$. In both stages, we need $n - 1$ operations. As the result, we obtain a sequence of length $2n - 2$.

Let T be an arbitrary spanning tree of G and v be its arbitrary vertex. Let v_1, \dots, v_n be an arbitrary order such that if $i > j$, then $d_T(v, v_i) < d_T(v, v_j)$ (hence $v_n = v$). We define $\text{suc}_T : v_i \mapsto v_j$ such that $j > i$ and v_i and v_j are connected in T for $i \in 1, \dots, n$, and $\text{suc}(v_n) = v_n$. Since T is a tree, function is well-defined. Moreover, $\mathbf{U}_T \subseteq \mathbf{U}_G$.

Suppose $|\varphi\rangle$ is given. The algorithm goes as follows. At each step $i \in [n - 1]$ we choose unitary operation $U_i \in \mathbf{U}_T$ changing superposition $\alpha_{v_i}|v_i\rangle + \alpha_{\text{suc}(v_i)}|\text{suc}(v_i)\rangle$ into $\beta_{\text{suc}(v_i)}|\text{suc}(v_i)\rangle$ such that $|\beta_{\text{suc}(v_i)}|^2 = |\alpha_{v_i}|^2 + |\alpha_{\text{suc}(v_i)}|^2$ and acting trivially on the other canonical states. Simple induction shows that, after $n - 1$ steps, we obtain state $|v\rangle$. The reversed method enables us to obtain state $|\psi\rangle$. \square

The above theorem shows that with $O(|V|)$ steps the arbitrary state can be obtained. For some graphs, we need much fewer steps. Even the algorithm from the proof can be optimised since in some cases the operations can be performed simultaneously. In the case of the complete graph, an arbitrary state can be obtained in one step. However, one should note that this bound cannot be decreased in general. This can be verified by analysing the segment graph.

3.3 Nontriviality of the operations

While it is very simple to construct proper stochastic operations preserving the graph structure, it is not the case for unitary operations. It is because of the fact, that construction of stochastic operations can be done independently for each column. In the case of unitary operation, we need to check whether columns are pair-ways orthogonal.

Suppose the graph in Fig. 2a is given. Let us start with state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$, where $\alpha, \beta, \gamma \neq 0$. Our goal is to find single graph-preserving unitary operation which will change the state into $e^{i\psi}|1\rangle$. Note that all unitary operations need to be of the form

$$\begin{bmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix}, \quad (6)$$

where mark ‘ \times ’ denotes the possibly nonzero value. In order to preserve unitarity of the matrix, the first and the last columns need to be orthogonal. Hence $\langle 0|A|1\rangle$ or $\langle 2|A|1\rangle$ equals 0. In this case the matrix preserves respectively $\alpha|0\rangle$ or $\gamma|2\rangle$ part of the state up to phase. The result can be generalized to the star graphs of the size at least 3. Therefore, it is impossible to obtain the goal. At the same time, it is easy to find a stochastic operation which changes arbitrary probability vector $[p_0, p_1, p_2]$ into $[0, 1, 0]$.

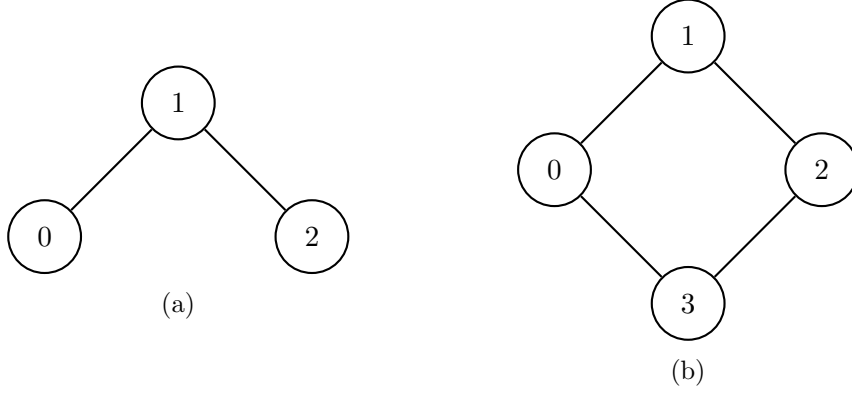


Figure 2: (a) An example of a graph for which it is impossible to find graph-preserving unitary operation changing state $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ to state $|1\rangle$. At the same time it is simple to find a stochastic operation which performs similar operation. (b) In this case it is possible to find an operation which changes the state $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ to $|1\rangle$, for $\alpha, \beta, \gamma \neq 0$.

The situation diametrically changes in the case of the graph in Fig. 2b. Suppose that we again start in state $|\varphi\rangle = r_\alpha e^{ik_\alpha}|0\rangle + r_\beta e^{ik_\beta}|1\rangle + r_\gamma e^{ik_\gamma}|2\rangle$, where all the parameters are nonnegative. Then for

$$V = \begin{bmatrix} -e^{i(k_\beta - k_\gamma + \psi)} r_\beta & e^{i(k_\alpha - k_\gamma + \psi)} r_\alpha & 0 & e^{-i(-k_\alpha + k_\gamma + \alpha + \pi)} r_\gamma \\ e^{-i(k_\alpha - k_\gamma + \psi)} r_\alpha & e^{-i(k_\beta - k_\gamma + \psi)} r_\beta & e^{-i\psi} r_\gamma & 0 \\ 0 & e^{i\psi} r_\gamma & -e^{i(k_\beta - k_\gamma + \psi)} r_\beta & e^{-i\alpha} r_\alpha \\ -e^{-i(k_\alpha - k_\gamma - \alpha)} r_\gamma & 0 & e^{i\alpha} r_\alpha & e^{-i(k_\beta - k_\gamma + \psi)} r_\beta \end{bmatrix} \quad (7)$$

we have $V|\varphi\rangle = e^{i(k_\gamma - \psi)}|1\rangle$. Note that

$$V|3\rangle = -e^{-i(-k_\alpha + k_\gamma + \alpha)} r_\gamma|0\rangle + e^{-i\alpha} r_\alpha|2\rangle + e^{-i(k_\beta - k_\gamma + \psi)} r_\beta|3\rangle, \quad (8)$$

the amplitudes are fixed and we can only change the local phase.

4 Quantum controlled Cop and Robber game

Taking into account the discussion in Section 3 we can now define and analyse another quantized version of the Cop and Robber game. Using graph-preserving quantum operations, we generalise the available strategies into quantum controlled graph-preserving unitary operations. Such game differs significantly from the previously mentioned ones, at least in the sense of probability of winning available for the Cop. We show that a quantum controlled graph preserving unitary operations generalises the original graph-preserving unitary operations.

4.1 Model definition

Similarly to classically controlled quantum Cop and Robber game, each player has its own quantum system spanned by $\{|v\rangle : v \in V\}$. At the beginning, the Cop and the Robber, in that order, choose their initial states. However, in contrast to the previously defined game, the Robber can entangle arbitrarily with the Cop at the beginning, by performing the controlled operation. In this model, the players do not possess the knowledge about the global states and are allowed to perform quantum controlled graph-preserving unitary operations only. For this reason we call this model *quantum controlled Cop and Robber game*.

We define the set of allowed operations as follows. Let G be an arbitrary reflexive, connected graph. By \mathbf{U}_G we denote the set of graph-preserving unitary operations. Moreover, let $U : V \rightarrow \mathbf{U}_G$ be an arbitrary function. Then the operation

$$\tilde{U} = \sum_{v \in V} |v\rangle\langle v| \otimes U(v) \quad (9)$$

is quantum controlled unitary operation. Note that \tilde{U} is a unitary operation. We denote the set of all such operation as \mathbf{cU}_G . One can verify that

$$\mathbf{cU}_G = \mathbf{U}_{\underbrace{G + G + \dots + G}_{n \text{ times}}} \quad (10)$$

where $+$ denotes the disjoint union of the graphs.

In the quantum controlled Cop and Robber game, both players can perform arbitrary operations from \mathbf{cU}_G . The control part is performed on the opponent's system, whilst the unitary operations are performed on the player's system.

The probability of the Cop to win is the same as in the classically controlled quantum Cop and Robber game, *i.e.* if after t -th round the state of $\mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{C}}$ is $|S_t\rangle$, then

$$p_{\text{copwin}} = \sum_{v \in V} |(\langle v| \otimes \langle v|) |S_t\rangle|^2. \quad (11)$$

However, the formula cannot be simplified due to possible entanglement between players' systems. The representation of the game in the form of a quantum circuit is presented in Fig. 3. Note that in this game both players do not know the current global state. For this reason both players can prepare the strategies before the game.

The quantum-controlled game expands the classically controlled one in the sense of the possible operations. Let $U \in \mathbf{U}_G$. Then

$$\tilde{U} = \sum_{v \in V} |v\rangle\langle v| \otimes U = \left(\sum_{v \in V} |v\rangle\langle v| \right) \otimes U = \mathbf{I} \otimes U. \quad (12)$$

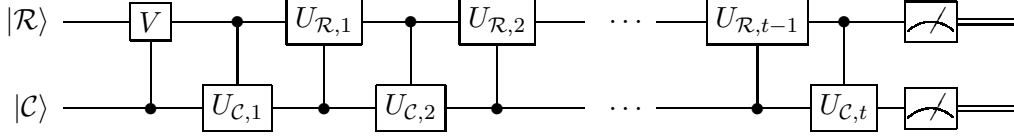


Figure 3: Quantum controlled quantum Cop and Robber game. The Cop chooses his initial state first, then the Robber chooses his state. The vertical line denotes quantum control, since both the Robber and the Cop know their and their opponent current state. Measurement is performed in basis $\{|v\rangle : v \in V\}$. $V \in \mathbf{U}(\mathcal{H}_{\mathcal{R}})$ is arbitrary unitary gate.

4.2 Non-triviality of the game

The crucial difference between classically controlled and quantum controlled versions of the quantum Cop and Robber game is observed in the possible set of strategies. The ability to introduce entanglement between the systems enables the Cop to win in one step for a large class of graphs.

Theorem 3. *Arbitrary graph which contains a universal vertex is 1-copwin in quantum controlled Cop and Robber game.*

Proof. The game goes as follows. The Cop starts in a universal vertex $|v_C\rangle$. Since the Cop starts in the canonical state, the Robber can only start in an arbitrary, separated state $|R\rangle = \sum_{v \in V} \alpha_v |v\rangle$.

Now we define strategy for the Cop. Let $U_{v \leftrightarrow v'}$ denote a transposition matrix, *i.e.*

$$U_{v \leftrightarrow v'} = |v\rangle\langle v'| + |v'\rangle\langle v| + \sum_{w \neq v, v'} |w\rangle\langle w|. \quad (13)$$

If v, v' are connected in G , then $U_{v \leftrightarrow v'} \in \mathbf{U}_G$. Obviously, if v is a universal vertex then for an arbitrary vertex v' the operation preserves the graph structure. The Cop chooses the operation

$$\tilde{U}_C = \sum_{v \in V} |v\rangle\langle v| \otimes U_{v \leftrightarrow v_C}. \quad (14)$$

The state changes into

$$\tilde{U}_C = \left(\sum_{v \in V} |v\rangle\langle v| \otimes U_{v \leftrightarrow v_C} \right) |R\rangle \otimes |v_C\rangle = \sum_{v \in V} \alpha_v |v\rangle \otimes |v\rangle. \quad (15)$$

By applying Eq. 11 we obtain the result. \square

Note that with the above strategy the Cop wins with probability one in a single step. The situation differs significantly in comparison both to the open probabilistic Cop and Robber and the classically controlled quantum Cop and Robber games. In these cases, the Cop wins with probability $\frac{1}{n}$ only.

Because of the above and the result from [25] we have the following corollary.

Corollary 1. *Almost all classical copwin graphs are 1-copwin in the quantum controlled Cop and Robber game.*

On the other hand it can be seen that the Cop cannot win in the general case. Let us use a cycle graph C_4 as an example. The Cop chooses $\sum_{i=0}^3 \alpha_i |i\rangle$ as the initial state. Then, it is optimal for the Robber to choose state $\sum_{i=0}^3 \alpha_i |i, i \oplus 2\rangle$. Simple analysis shows that for an arbitrary Cop's strategy there always exists a Robber's strategy such that the state before the Cop's move is of the form $\sum_{i=0}^3 \beta_i |i, i \oplus 2\rangle$. Hence, the Cop cannot win the game with nonzero probability. This example shows that the probability of winning depends on the graph.

4.3 Local versus non-local game

Let us now consider the following unfair game. The Cop is allowed to perform operations from \mathbf{cU}_{C_4} , while the Robber can only perform local operations from \mathbf{U}_{C_4} . Suppose the Cop starts in vertex $\frac{1}{2} \sum_{i=0}^3 |i\rangle$. Suppose the Robber chooses an arbitrary state $\sum_{i=0}^3 \alpha_i |i\rangle$. Then the Cop chooses the controlled operations described in Sec. 3. The state is now of the form

$$\sum_{i=0}^3 \sqrt{\frac{3}{4}} \alpha_i |i, i\rangle + \sum_{i,j=0:i \neq j}^3 \gamma_{i,j} |i, j\rangle, \quad (16)$$

where $\gamma_{i,j}$ comes from the performed operation. Note that

$$p_{\text{copwin}} = \sum_{i=0}^3 \frac{3}{4} |\alpha_i|^2 = \frac{3}{4}. \quad (17)$$

Comparing to the classically-controlled quantum Cop and Robber game, where the Cop can achieve probability $\frac{1}{4}$ at most, the Cop has much better possibilities.

5 Concluding remarks

In this paper we have introduced the quantum versions of the Cop and Robber game. As a tool we use unitary operations preserving the structure of the graph. We show that we can prepare an arbitrary state using such operations for arbitrary connected, reflexive digraph which contains a spanning undirected tree. Moreover, we have shown that for arbitrary initial and resulting states we need a sequence of $2n - 2$ operations at most, which is tight in the sense of complexity. We have also proposed a simple algorithm for obtaining such sequence.

Using the introduced operations, we quantized the Cop and Robber game. We propose two different quantum models. The classically controlled quantum Cop and Robber game trivialises in the sense of winning strategies and in this case both the Cop and the Robber can choose equal superposition as the initial state and achieve Nash equilibrium. Hence, the probability of winning for the Cop depends only on the vertex set size and

not on any other properties of the graph or the evolution model. In that sense, the game is similar to open probabilistic Cop and Robber game. Moreover, we show that both the classically controlled quantum Cop and Robber game and the open probabilistic Cop and Robber game expand the original game in the sense of available strategies. In the case of quantum controlled Cop and Robber game, we allow both players to perform quantum controlled operations, preserving the graph structure. We argue that the game differs significantly from both previously defined models in the sense of winning strategy. By this, we show that the classical control differs significantly from the quantum control. Moreover, we show that the quantum controlled Cop and Robber game expands the original game in the sense of available strategies. Unfortunately, we have not found any dependence between graph-preserving stochastic and unitary operations.

We have also analysed the case of unfair games. We show that the strategies available in the open probabilistic and classically controlled quantum Cop and Robber games are stronger than in the case of deterministic games. We also demonstrate that the strategies available in the quantum controlled Cop and Robber game are stronger than in the classically controlled quantum Cop and Robber game. While in the first one, the result of our analysis is applied to an arbitrary undirected graph, the latter was shown by offering some examples.

The results presented in this paper may be analysed in different directions. First, further analysis of the quantum controlled Cop and Robber game can be made. Moreover, the various classical generalisations of the original Cop and Robber game can be applied to the quantized version. Further analysis may provide new information concerning the quantum controlled operations and provide more insight into the differences between the classical and the quantum versions of pursuit-evasion games.

Acknowledgements

This work has been supported by the Polish National Science Centre under the project number 2011/03/D/ST6/00413. Part of this work was supported by COST Action CA15220 Quantum Technologies in Space. Authors would like to thank M. Ostaszewski for interesting remarks concerning quantum games on directed graphs.

References

- [1] J. A. Miszczak, “High-level structures for quantum computing,” *Synthesis Lectures on Quantum Computing*, vol. 4, no. 1, pp. 1–129, 2012. DOI: 10.2200/S00422ED1V01Y201205QMC006.
- [2] A. Quilliot, *Jeux et pointes fixes sur les graphes*. PhD thesis, Ph. D. Dissertation, Université de Paris VI, 1978.
- [3] R. Nowakowski and P. Winkler, “Vertex-to-vertex pursuit in a graph,” *Discrete Mathematics*, vol. 43, no. 2-3, pp. 235–239, 1983. DOI: 10.1016/0012-365X(83)90160-7.

- [4] P. Bernhard, A.-L. Colomb, and G. Papavassilopoulos, “Rabbit and hunter game: Two discrete stochastic formulations,” *Computers & Mathematics with Applications*, vol. 13, no. 1-3, pp. 205–225, 1987. DOI: 10.1016/0898-1221(87)90105-2.
- [5] G. Konstantinidis and A. Kehagias, “Simultaneously moving cops and robbers,” *Theoretical Computer Science*, vol. 645, pp. 48–59, 2016. DOI: 10.1016/j.tcs.2016.06.039.
- [6] A. Bonato, D. Mitsche, X. Pérez-Giménez, and P. Prałat, “A probabilistic version of the game of zombies and survivors on graphs,” *Theoretical Computer Science*, vol. 655, pp. 2–14, 2016. DOI: 10.1016/j.tcs.2015.12.012.
- [7] S. Fitzpatrick, J. Howell, M. Messinger, and D. Pike, “A deterministic version of the game of zombies and survivors on graphs,” *Discrete Applied Mathematics*, vol. 213, pp. 1–12, 2016. DOI: 10.1016/j.dam.2016.06.019.
- [8] C. H. Fitzgerald, “The princess and monster differential game,” *SIAM Journal on Control and Optimization*, vol. 17, no. 6, pp. 700–712, 1979. DOI: 10.1137/0317049.
- [9] S. Alpern, R. Fokkink, R. Lindelauf, and G.-J. Olsder, “The princess and monster game on an interval,” *SIAM Journal on Control and Optimization*, vol. 47, no. 3, pp. 1178–1190, 2008. DOI: 10.1137/060672054.
- [10] M. Boyer, S. El Harti, A. El Ouarari, R. Ganian, T. Gavenčiak, G. Hahn, C. Moldenauer, I. Rutter, B. Thériault, and M. Vatshelle, “Cops-and-robbers: remarks and problems,” *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 85, 2013.
- [11] P. Frankl, “Cops and robbers in graphs with large girth and Cayley graphs,” *Discrete Applied Mathematics*, vol. 17, no. 3, pp. 301–305, 1987. DOI: 10.1016/0166-218X(87)90033-3.
- [12] T. H. Chung, G. A. Hollinger, and V. Isler, “Search and pursuit-evasion in mobile robotics,” *Autonomous robots*, vol. 31, no. 4, p. 299, 2011. DOI: 10.1007/s10514-011-9241-4.
- [13] A. Bonato and B. Yang, “Graph searching and related problems,” in *Handbook of Combinatorial Optimization*, pp. 1511–1558, Springer, 2013. DOI: 10.1007/978-1-4419-7997-1_76.
- [14] A. Bonato, P. Prałat, and C. Wang, “Pursuit-evasion in models of complex networks,” *Internet Mathematics*, vol. 4, no. 4, pp. 419–436, 2007. DOI: 10.1080/15427951.2007.10129149.
- [15] A. Bonato and R. J. Nowakowski, *The game of cops and robbers on graphs*, vol. 61. American Mathematical Society Providence, 2011. ISBN: ISBN 978-0-8218-5347, DOI: 10.1090/stml/061.

- [16] E. W. Piotrowski and J. Ślaskowski, “An invitation to quantum game theory,” *International Journal of Theoretical Physics*, vol. 42, no. 5, pp. 1089–1099, 2003. DOI: 10.1023/A:1025443111388.
- [17] J. Eisert, M. Wilkens, and M. Lewenstein, “Quantum games and quantum strategies,” *Physical Review Letters*, vol. 83, no. 15, p. 3077, 1999. DOI: 10.1103/PhysRevLett.83.3077.
- [18] L. Pawela and J. Ślaskowski, “Cooperative quantum parrondos games,” *Physica D: Nonlinear Phenomena*, vol. 256, pp. 51–57, 2013. DOI: 10.1016/j.physd.2013.04.010.
- [19] D. A. Meyer, “Quantum strategies,” *Physical Review Letters*, vol. 82, pp. 1052–1055, Feb 1999. DOI: 10.1103/PhysRevLett.82.1052.
- [20] W. Hao, D. Chao-Wei, and F. Bao-Fu, “A novel multi pursuers-one evader game based on quantum game theory,” *Information Technology Journal*, vol. 12, no. 12, p. 2358, 2013. DOI: 10.3923/itj.2013.2358.2365.
- [21] J. A. Mischczak and P. Sadowski, “Quantum network exploration with a faulty sense of direction,” *Quantum Information & Computation*, vol. 14, no. 13-14, pp. 1238–1250, 2014.
- [22] P. Sadowski, J. A. Mischczak, and M. Ostaszewski, “Lively quantum walks on cycles,” *Journal of Physics A: Mathematical and Theoretical*, vol. 49, no. 37, p. 375302, 2016. DOI: 10.1088/1751-8113/49/37/375302.
- [23] A. Condon, “On algorithms for simple stochastic games,” in *Advances in computational complexity theory*, pp. 51–72, 1990.
- [24] A. Montanaro, “Quantum walks on directed graphs,” *Quantum Information & Computation*, vol. 7, no. 1, pp. 93–102, 2007.
- [25] A. Bonato, G. Kemkes, and P. Prałat, “Almost all cop-win graphs contain a universal vertex,” *Discrete Mathematics*, vol. 312, no. 10, pp. 1652–1657, 2012. DOI: 10.1016/j.disc.2012.02.018.